does not, in fact, contain any of the omissions noted in the text of [4]). The presence of a finite number of formal invariants of the second order non-Hamiltonian systems with resonances was established earlier [5]. The same aspect was studied for the multidimensional systems by the author in [6] and (simultaneously and independently) in [7].

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# EVENTUAL STABILITY OF DIFFERENTIAL SYSTEMS OF NEUTRAL TYPE

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S. N. SOROKIN
(Moscow)

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For differential systems of neutral type we examine one of the formulations of the finite-time interval stability problem, i.e., technical stability. By the Liapunov-Krasovskii method [1 — 3] we obtain sufficient conditions for technical stability and for the so-called contracting technical stability. Similar investigations for ordinary differential equations were carried out in [4] and for equations with a lagging argument, in [5, 6].

**1.** We are given a system of differential equations

$$\frac{d}{dt} D(x_t(\theta), t) = f(x_t(\theta), t), \quad D(x_t(\theta), t) \equiv x(t) - g(x_t(\theta), t) \tag{1.1}$$

$$g(x(\theta), t) \equiv \int_{-\tau}^{0} [d_\theta \mu(\theta, t)]\, x(\theta)$$

Here the vector function $x_t(\theta) \equiv x(t+\theta)$ belongs for all $t \geqslant 0$ to the space $C_0 \equiv C([-\tau, 0], R^n)$ with the norm $\| x(\theta) \| = \sup (\mid x_i(\theta) \mid$ for $-\tau \leqslant \theta \leqslant 0$, $i = 1, 2, \ldots, n)$; $\mu(\theta, t)$ is an $(n \times n)$-matrix of functions continuous in $t \in [0, \infty)$ and of bounded variation in $\theta$, for which a continuous function $l_0(s)$, nondecreasing in $s \in [0, \tau]$ , $l_0(0) = 0$, exists such that

$$\left| \int_{-s}^{0} [d_\theta \mu(\theta, t)] x(\theta) \right| \leqslant l_0(s) \sup_{-s \leqslant \theta \leqslant 0} \| x(\theta) \|$$
$$\forall t \in [0, \infty), \ \forall x(\theta) \in C_0$$

The continuous $n$-dimensional vector functional $f(x(\theta), t)$ is chosen in such a way that the solution of system (1.1), defined by the initial function $x_{t_0}(\theta) = \varphi_{t_0}(\theta)$, $-\tau \leqslant \theta \leqslant 0$ exists and is unique [7].

Definition 1. System (1.1) is said to be $(\alpha, A, t_0, T)$-stable $(\alpha < A)$ if the solution $x_t(\theta)$ defined by the initial function $\| x_{t_0}(\theta) \| \leqslant \alpha$ satisfies the inequality $\| x_t(\theta) \| < A$, $\forall t \in [t_0, t_0 + T)$.

Definition 2. System (1.1) is said to be $(\alpha, A, B, t_0, T)$-contractively stable $(B < \alpha < A)$ if it is $(\alpha, A, t_0, T)$-stable and if for the solution $x_t(\theta)$ defined by the initial function $\| x_{t_0}(\theta) \| \leqslant \alpha$ we can find an instant $t_1 \in (t_0, t_0 + T)$, $t_1 = t_1(t_0, x_{t_0}(\theta))$, such that $\| x_t(\theta) \| < B$, $\forall t \in [t_1, t_0 + T)$.

Definition 3. System (1.1) is said to be $(\alpha, A, t_0, T)$-unstable $(\alpha < A)$ if there exists even one trajectory $x_t^*(\theta)$ defined by the initial function $\| x_{t_0}^*(\theta) \| \leqslant \alpha$, for which the relation $\| x_{t_1}^*(\theta) \| = A$ holds at some instant $t_1 \in (t_0, t_0 + T)$.

The basic aim of the paper is to find the relations between the finite allowances for the input $\alpha$ and the output $A$, $B$, the time constraints on $t_0$ and $T$ and the parameters of the considered system (1.1), which would guarantee stability in a finite time interval. It is important to note that some of the constraints imposed upon the parameters of system (1.1) in order to ensure asymptotic Liapunov stability are insufficient for technical stability. On the other hand, the introduced stability (contracting stability) can hold even when the parameters of the system (1.1) do not ensure its Liapunov stability.

2. As a preliminary we consider a functional-difference operator $D(x(\theta), t)$ and the corresponding functional-difference system

$$x(t) - g(x_t(\theta), t) = h(t), \quad t \geqslant t_0, \quad x_{t_0}(\theta) = \varphi_{t_0}(\theta) \tag{2.1}$$
$$h(t) \in C_0^* \equiv C([0, \infty), R^n)$$

whose properties we use in the proof of the engineering stability theorem.

Definition 4. Operator $D$ is said to be exponentially increasing uniformly with respect to space $C_0^*$ if constants $K_1 > 0$, $K_2 > 0$ and $a$ exist such that for any functions $\varphi_{t_0}(\theta) \in C_0$ and $h(t) \in C_0^*$ and for an arbitrary instant $t_0 \in [0, \infty)$ the continuous solution $x_t(\theta)$ of system (2.1) satisfies the relation

$$\| x_t(\theta) \| \leqslant K_1 \| \varphi_{t_0}(\theta) \| \exp\{a(t - t_0)\} + K_2 H(t - t_0) \sup_{t_0 \leqslant u \leqslant t} \| h(u) \|, \quad t \geqslant t_0 \tag{2.2}$$

where $H(r)$ is the exponential function $e^{ar}$ if $a > 0$, is the linear function $b + cr$ if $a = 0$, and is the constant $d$ if $a < 0$.

Sufficient conditions for operator $D$ to be exponentially increasing uniformly with respect to $C_0^*$ are given by

Lemma 1. If the matrix of the functions $\mu(\theta, t)$ is such that for some $\varepsilon \in (0, \tau]$

$$\int_{-\tau}^{0} [d_\theta \mu(\theta, t)] x(\theta) \equiv \int_{-\tau}^{-\varepsilon} [d_\theta \mu(\theta, t)] x(\theta) \tag{2.3}$$

then operator $D$ is exponentially increasing.

Proof. We have

$$\left|\int_{-\tau}^{-\varepsilon}[d_\theta\mu(\theta,t)]\,x_t(\theta)\right|\leqslant L_0\|x_t(\theta)\|,\qquad -\tau\leqslant\theta\leqslant-\varepsilon,\qquad L_0=\text{const}$$

At first we assume that $L_0<1$. Using the step method, we find an estimate for the continuous solution $x_t(\theta)$ of system (2.1). As steps we take the semi-intervals $[t_0+(k-1)\varepsilon,\ t_0+k\varepsilon)$, $k=1,2,\ldots$. We take the constant $N_0=\tau/\varepsilon$ if $\tau/\varepsilon$ is an integer and $N_0=[\tau/\varepsilon]+1$ in the remaining cases ($[a]$ is the integer part of number $a$). This constant characterizes the duration of the aftereffect. After the $N_0$-th step the aftereffect of the process' states preceding the instant $t_0$ comes to an end. Estimating stepwise, at the $(N_0(k-1)+j)$-th semi-interval $[t_0+(N_0(k-1)+j-1)\varepsilon,\ t_0+(N_0(k-1)+j)\varepsilon)$, $j=1,2,\ldots,N_0$, $k=1,2,\ldots$, we obtain

$$\|x_{t(kj)}(\theta)\|\leqslant L_0^{k-1}\|\varphi_{t_0}(\theta)\|+\sum_{l=0}^{N_0(k-1)+j-1}L_0^lS_{t_0}(h),\qquad S_{t_0}(h)\equiv\sup_{t_0\leqslant u\leqslant t}\|h(u)\|$$

It is easy to see that the inequality

$$L_0^k\leqslant\exp\left\{\left(\left[\frac{t-t_0}{\varepsilon N_0}\right]+1\right)\ln L_0\right\}\leqslant\exp\left\{\frac{\ln L_0}{\varepsilon N_0}(t-t_0)\right\}$$

is valid on each of the semi-intervals $t_0+(k-1)N_0\varepsilon\leqslant t<t_0+kN_0\varepsilon$, $k=1,2,\ldots$ This relation is not violated as $k\to\infty$ (the possibility of an unbounded stepwise estimation process follows from condition (2.3)). Therefore, for $t\geqslant t_0$

$$\|x_t(\theta)\|\leqslant L_0^{-1}\|\varphi_{t_0}(\theta)\|\exp\left\{\frac{\ln L_0}{\varepsilon N_0}(t-t_0)\right\}+\frac{1}{1-L_0}S_{t_0}(h)$$

Now let $L_0\geqslant1$. As above, estimating stepwise, at the $k$-th semi-interval $[t_0+(k-1)\varepsilon,\ t_0+k\varepsilon)$, $k=1,2,\ldots$, we obtain

$$\|x_{t(k)}(\theta)\|\leqslant L_0^k\|\varphi_{t_0}(\theta)\|+\sum_{l=0}^{k-1}L_0^lS_{t_0}(h)$$

Using the obvious relation

$$k-1=\left[\frac{t-t_0}{\varepsilon}\right]\leqslant\frac{t-t_0}{\varepsilon}\tag{2.4}$$

on each of the semi-intervals being examined we obtain

$$L_0^k\leqslant L_0\exp\left\{\frac{\ln L_0}{\varepsilon}(t-t_0)\right\},\quad k=1,2,\ldots\tag{2.5}$$

Relations (2.4) and (2.5) are not violated as $k\to\infty$. Therefore, for $t\geqslant t_0$

$$\|x_t(\theta)\|\leqslant L_0\left[\|\varphi_{t_0}(\theta)\|+\frac{1}{L_0-1}S_{t_0}(h)\right]\exp\left\{\frac{\ln L_0}{\varepsilon}(t-t_0)\right\},\quad L_0>1$$

$$\|x_t(\theta)\|\leqslant\|\varphi_{t_0}(\theta)\|+\left[1+\frac{1}{\varepsilon}(t-t_0)\right]S_{t_0}(h),\quad L_0=1$$

The lemma is proved.

Note 1. If operator $D$ is exponentially increasing uniformly with respect to $C_0{}^*$ and if $a<0$, then positive constants $K_i^\circ$, $i=1,2,3$ exist such that the continuous solution of system (2.1) satisfies the inequality

$$\|x_t(\theta)\|\leqslant[K_1^\circ\|\varphi_{t_0}(\theta)\|+K_2^\circ S_{t_0}(h)]\exp\{a(t-t_0)\}+K_3^\circ S_s(h)\tag{2.6}$$

for any $s \in [t_0, \infty)$ and $t \geqslant s$.

Together with system (2.1) we consider the system

$$D(x_t(\theta), t) = D(\varphi_{t_0}(\theta), t_0) + p(t) - p(t_0) \tag{2.7}$$
$$t \geqslant t_0, \; x_{t_0}(\theta) = \varphi_{t_0}(\theta)$$

where $p(t) \in C_0^*$.

Definition 5 [3]. We assume that $C_0^{**} \subset C_0^*$. Operator $D$ is said to be uniformly stable with respect to $C_0^{**}$ if positive constants $M_1$ and $M_2$ exist such that the solution $x_t(\theta)$ of system (2.7) satisfies the relation

$$\|x_t(\theta)\| \leqslant M_1 \|\varphi_{t_0}(\theta)\| + M_2 \sup_{t_0 \leqslant u \leqslant t} \|p(u) - p(t_0)\|, \quad t \geqslant t_0$$

for any functions $\varphi_{t_0}(\theta) \in C_0$ and $p(t) \in C_0^{**}$ and for any arbitrary instant $t_0 \in [0, \infty)$

It was shown in [3] that if operator $D$ is independent of $t$, then from the condition of uniform stability it follows that the roots $\lambda$ of the equation

$$\det\left[E - \int_{-\tau}^{0} [d\mu(\theta)]\lambda^{\theta}\right] = 0$$

are not greater than $1 - \delta$, $\delta > 0$ in absolute value. The converse has not been proved.

Definition 6. We say that the system of functions $q_j(t)$, $j = 1, 2, \ldots, k$ is united by means of function $q^{\circ}(t)$, with union coefficients $1 \leqslant m_1 < m_2 < \ldots < m_k$ (the $m_j$ are integers), if $q_j(t) = q_{m_j}^{\circ}(t)$, where $q_{m_j}^{\circ}(t) \equiv q^{\circ}(q^{\circ}(\ldots(q^{\circ}(t))\ldots))$ is the $m_j$-fold iteration of operation $q^{\circ}(t)$.

When $q_j = t - \tau_j$ (the $\tau_j$ are constants, $1 \leqslant j \leqslant k$), the union by means of the function $q^{\circ}(t) = t - \Delta_0$ is equivalent to the commensurability of the constants $\tau_j$ with the largest general measure $\Delta_0$.

Lemma 2. Let operator $D$ have the form

$$D(x_t(\theta), t) = x(t) - \sum_{i=1}^{M} P_i(t)\, x(q_i(t)), \quad q_i(t) \geqslant t - \tau$$

where $P_i(t)$ are $(n \times n)$-matrices of continuous functions and the function $q_i(t)$ $i = 1, 2, \ldots, M$ united by means of a continuous function $q^{\circ}(t)$ increasing for $t \geqslant t_0$ and satisfying the condition $t - q^{\circ}(t) \geqslant d_0 > 0$, $d_0 = \text{const}$. If the roots $\lambda(t)$ of the equation

$$\det\left[E - \sum_{i=1}^{M} P_i(t)\lambda(t)^{-m_i}\right] = 0$$

do not exceed $1 - \delta$ $(\delta > 0)$ in absolute value, then operator $D$ is uniformly stable with respect to $C_0^*$.

Lemma 2 is the natural generalization of the similar lemma in [3] to the case of a variable lag.

Lemma 3 [3]. If operator $D$ is uniformly stable with respect to $C_0^*$, then positive constants $a^*$, $K_1^*$, $K_2^*$ and $K_3^*$ exist such that the continuous solution $x_t(\theta)$ of system (2.1) satisfies the inequality

$$\|x_t(\theta)\| \leqslant [K_1^* \|\varphi_{t_0}(\theta)\| + K_2^* S_{t_0}(h)] \exp\{-a^*(t - t_0)\} + K_3^* S_{t_0}(h)$$

for any functions $\varphi_{t_0}(\theta) \in C_0$ and $h(t) \in C_0{}^*$ and for an arbitrary instant $t_0 \in [0, \infty)$.

Here Note 1 on the peculiarities of the constants $K_i{}^*$, $i = 1, 2, 3$ remains in force.

In a number of cases Lemmas 2 and 3 permit us to improve the estimate of the solution of system (2. 1) obtained in Lemma 1.

**3.** We make use of the Liapunov-Krasovskii method [1—3] to answer the question of eventual stability of systems of neutral type (1. 1). We examine the functionals $V(x(\theta), t) \equiv V(x(\theta); D(x(\theta), t); t)$, continuous in their arguments, defined on the continuous functions $x(t) \in C_0$ and translating a bounded set of elements of space $C_0$ into a bounded set of elements of space $R^1$. By the upper right-hand derivative of functional $V$ by virtue of system (1. 1), we imply that

$$\overline{V}_1{}^{\cdot} \equiv \overline{\lim_{\Delta t \to 0+}}\, \frac{1}{\Delta t}\left[V(x^*_{t+\Delta t}(\theta),\ t+\Delta t) - V(x(\theta), t)\right]$$

where $x^*_{t+\Delta t}(\theta)$ is the solution of system (1. 1), defined by the initial instant $t$ and the initial function $x_t{}^*(\theta) = x(\theta)$. The lower right derivative $V_1{}^{\cdot}$ of functional $V$ by virtue of system (1. 1) is defined similarly.

Theorem 1. Let operator $\overline{D}$ be exponentially increasing uniformly with respect to $C_0{}^*$ and norm $L(t)$ of this operator satisfy the relation

$$1^\circ.\ L(t) < (A_0 - K_1\alpha)\frac{1}{\alpha K_4} \equiv \frac{\Gamma_1}{\alpha},\quad \forall t \in [t_0, t_0+T)$$

$$A_0 \equiv \min\{Ae^{-aT};\ A\},\quad K_4 \equiv \max\{K_2;\ (b+cT)K_2;\ dK_2\}$$

If a functional $V(x(\theta), t)$ and an integrable function $\psi(t)$ exist such that

$$2^\circ.\ \overline{V}_1{}^{\cdot} < \psi(t),\quad \alpha \leqslant \|x(\theta)\| \leqslant A,\quad \|D(x(\theta), t)\| \leqslant \Gamma_1,\quad \forall t \in [t_0, t_0+T)$$

$$3^\circ.\ \int_{t_1}^{t_2}\psi(t)\,dt \leqslant \inf_{Q_1(t_2)} V(x(\theta), t_2) - \sup_{Q_2(t_1)} V(x(\theta), t_1);\quad \forall t_1, t_2 \in [t_0, t_0+T)$$

$$t_2 > t_1$$

$$Q_1(t) = \{x(\theta):\ \Gamma_1/L(t) \leqslant \|x(\theta)\| \leqslant A,\quad \|D(x(\theta), t)\| = \Gamma_1\}$$

$$Q_2(t) = \{x(\theta:\ \|x(t)\| = \alpha,\ \|D(x(\theta), t)\| \leqslant \alpha L(t)\},\quad \forall t \in [t_0, t_0+T)$$

then system (1. 1) is $(\alpha, A, t_0, T)$-stable.

Proof. Let $x_t(\theta)$ be a solution of system (1. 1), defined by the initial function $x_{t_0}(\theta)$ lying in the domain $\|x(\theta)\| \leqslant \alpha$. From condition 1° it follows that

$$\|D(x_{t_0}(\theta),\ t_0)\| \leqslant L(t_0)\|x_{t_0}(\theta)\| \leqslant \Gamma_1$$

Let us assume that

$$\|D(x_{t_2}(\theta), t_2)\| = \Gamma_1 \tag{3. 1}$$

for the first time at some instant $t_2 \in (t_0, t_0+T)$. If $\Gamma_1 > AL(t_2)$, then from (3. 1) it follows that $\|x_{t_2}(\theta)\| > A$ at the instant being examined. On the other hand, by assumption $\|D(x_t(\theta), t)\| \leqslant \Gamma_1$ for $t \in [t_0, t_2]$. According to (2. 2) this implies that $\|x_{t_2}(\theta)\| \leqslant A$. We have obtained a contradiction. Consequently, $\|D(x_{t_2}(\theta), t_2)\| < \Gamma_1$.

However, if $\Gamma_1 \leqslant A\ L(t_2)$, then, making use of condition 1°, from (3. 1) we obtain $\|x_{t_2}(\theta)\| \geqslant \Gamma_1 / L(t_2) > \alpha$. Therefore, an instant $t_1 < t_2$ exists for which

$\| x_{t_1}(\theta) \| = \alpha$ ($\| D(x_{t_1}(\theta), t_1) \| \leqslant L(t_1)\alpha$) and $\| x_t(\theta) \| > \alpha$ for $t_1 < t \leqslant t_2$. On the basis of relation (2.2) we conclude that $\alpha \leqslant \| x_t(\theta) \| \leqslant A$, $\forall t \in [t_1, t_2]$. Relations 2° and 3° are fulfilled in the domain being examined. Therefore,

$$\inf_{Q_1(t_2)} V(x(\theta), t_2) - \sup_{Q_2(t_1)} V(x(\theta), t_1) \leqslant V(x_{t_2}(\theta), t_2) - V(x_{t_1}(\theta), t_1) \leqslant$$
$$\int_{t_1}^{t_2} \bar{V}_1^{\cdot}(t)\,dt < \int_{t_1}^{t_2} \psi(t)\,dt \leqslant \inf_{Q_1(t_2)} V(x(\theta), t_2) - \sup_{Q_2(t_1)} V(x(\theta), t_1)$$

The relation obtained is a contradiction. Consequently, in this case too $\| D(x_{t_2}(\theta), t_2) \| < \Gamma_1$. Since the instant $t_2$ being examined is arbitrary, we conclude that $\| D(x_t(\theta), t) \| < \Gamma_1$ for all $t \in [t_0, t_0 + T)$. In accord with relation (2.2), the latter leads to the inequality $\| x_t(\theta) \| < A$ for $t \in [t_0, t_0 + T)$. The theorem is proved.

Theorem 2. If a functional $V(x(\theta), t)$, the integrable functions $\psi_i(t)$, $i = 1, 2, 3$ and the constants $\beta$ and $\gamma$ ($0 < \beta < B$ and $0 \leqslant \gamma \leqslant \beta$) are such that conditions 1°—3° of Theorem 1 and the conditions:

4°. the norm $L(t)$ of operator $D$ satisfies the inequality

$$\beta L(t) \leqslant \frac{B_0 - K_1\beta}{K_4} \equiv \Gamma_2, \quad B_0 \equiv \min\{Be^{-\alpha T};\ B\}$$

5°. $\bar{V}_1^{\cdot} < \psi_2(t), \quad \beta \leqslant \| x(\theta) \| \leqslant A, \quad \| D(x(\theta), t) \| \leqslant \Gamma_1$

6°. $\displaystyle\int_{t_0}^{t_0+T} \psi_2(t)\,dt \leqslant \inf_{Q_3(t_0+T)} V(x(\theta), t_0 + T) - \sup_{Q_4(t_0)} V(x(\theta), t_0)$

$$Q_3(t) = \{x(\theta):\ \beta \leqslant \| x(\theta) \| \leqslant A,\ \| D(x(\theta), t) \| \leqslant \Gamma_1\}$$
$$Q_4(t) = \{x(\theta):\ \beta \leqslant \| x(\theta) \| \leqslant \alpha,\ \| D(x(\theta), t) \| \leqslant \alpha L(t)\}$$

7°. $\bar{V}_1^{\cdot} < \psi_3(t), \quad \gamma \leqslant \| x(\theta) \| \leqslant B, \quad 0 \leqslant \| D(x(\theta), t) \| \leqslant \Gamma_2$

8°. $\displaystyle\int_{t_1}^{t_2} \psi_3(t)\,dt \leqslant \inf_{Q_5(t_2)} V(x(\theta), t_2) - \sup_{Q_6(t_1)} V(x(\theta), t_1), \quad \forall t_1, t_2 \in [t_0, t_0 + T)$

$t_2 > t_1$

$$Q_5(t) = \{x(\theta):\ \Gamma_2 / L(t) \leqslant \| x(\theta) \| \leqslant B,\ \| D(x(\theta), t) \| = \Gamma_2\}$$
$$Q_6(t) = \{x(\theta):\ \gamma \leqslant \| x(\theta) \| \leqslant \beta,\ 0 \leqslant \| D(x(\theta), t) \| \leqslant \beta L(t)\}$$

are fulfilled for all $t \in [t_0, t_0 + T)$, then system (1.1) is $(\alpha, A, B, t_0, T)$-contractively stable.

Proof. Let $x_t(\theta)$ be a solution of system (1.1), defined by the initial function $x_{t_0}(\theta)$ and located in the domain $\beta \leqslant \| x(\theta) \| \leqslant \alpha$. Using Theorem 1, we obtain that $\| x_t(\theta) \| < A$ on the finite time interval $[t_0, t_0 + T)$.

We now assume that the solution being investigated remains in the domain $\beta \leqslant \| x(\theta) \| \leqslant A$. Using conditions 5° and 6°, we obtain

$$\inf_{Q_3(t_0+T)} V(x(\theta), t_0 + T) - \sup_{Q_4(t_0)} V(x(\theta), t_0) \leqslant \tag{3.2}$$
$$V(x_{t_0+T}(\theta), t_0 + T) - V(x_{t_0}(\theta), t_0) \leqslant$$

$$\int_{t_0}^{t_0+T} \bar{V}_1^{\cdot}(t)\,dt < \int_{t_0}^{t_0+T} \psi_2(t)\,dt \leqslant \inf_{Q_3(t_0+T)} V(x(\theta),\ t_0+T) - \sup_{Q_4(t_0)} V(x(\theta),\ t_0)$$

From the inconsistency of the inequality it follows that the assumption made is incorrect, and, consequently, the solution is found to be in the domain $\| x(\theta) \| < \beta$ at some instant $t^* \in (t_0,\ t_0 + T)$, and by condition 4°

$$\| D(x_{t^*}(\theta),\ t^*) \| \leqslant L(t^*) \| x_{t^*}(\theta) \| < \Gamma_2$$

It remains to show that $\| x_t(\theta) \| < B$ for $t \in [t^*,\ t_0 + T)$.

Let us consider the case when $\Gamma_2 = \beta L(t)$ $(0 \leqslant \gamma < \beta)$ for at least one instant $t \in [t^*,\ t_0 + T)$. We assume that

$$\| D(x_{t_4}(\theta), t_4) \| = \Gamma_2 \tag{3.3}$$

for the first time at some instant $t_4$. It is easy to show that relation (3.3) is impossible when $\Gamma_2 > BL(t_4)$. However, if $\Gamma_2 \leqslant BL(t_4)$, then, making use of condition 4°, from relation (3.3) we obtain $\| x_{t_4}(\theta) \| \geqslant \beta$. Consequently, instants $t_1$ and $t_3$ $(t_0 \leqslant t_1 < t_3 \leqslant t_4)$, exist for which $\gamma \leqslant \| x_t(\theta) \| \leqslant \beta$ for $t \in [t_1,\ t_3]$ and $\beta \leqslant \| x_t(\theta) \| \leqslant B$ for $t \in [t_3,\ t_4]$. Having fixed the instant $t_2 \in [t_1,\ t_3]$, $t_2 < t_4$, from relations 7° and 8° we have a contradictory inequality similar to (3.2). Consequently, in this case too $\| D(x_{t_4}(\theta),\ t_4) \| < \Gamma_2$.

Using the arbitrariness of the instant $t_4$ being examined, it is easy to obtain from relations (2.2) and 4° that $\| x_t(\theta) \| < B$ for $t \in [t^*,\ t_0 + T)$. The proof is similar for the case $\beta L(t) < \Gamma_2$ $(\gamma = \beta)$. However, if the initial function $x_{t_0}(\theta)$ defining the solution of the system under investigation lies in the domain $\| x(\theta) \| < \beta$, then, by applying the method of proof presented, we can show that the solution being examined is to be found in the domain $\| x(\theta) \| < B < A$ for all $t \in [t_0,\ t_0 + T)$; whence it follows that $\| x_t(\theta) \| < B$ for $t \in [t^*,\ t_0 + T)$. The theorem is proved.

Note 2. Theorems 1 and 2 remain in force for an operator $D$ uniformly stable with with respect to space $C_0^*$ if we set

$$\Gamma_1^* \equiv \frac{A - K_1^*\alpha}{K_2^* + K_3^*}, \quad \Gamma_2^* \equiv \frac{B - K_1^*\beta}{K_2^* + K_3^*}$$

Corollary 1. For an operator $D$ uniformly stable with respect to space $C_0^*$, let there exist the functions $u_i(r,\ t)$ continuous in their arguments, $i = 1,\ 2,\ 3$ $(u_i(r,\ t))$, $i = 1,\ 2$ are nondecreasing in $r$ for $r > 0$ and $u_3(r,\ t)$ is nonpositive and nonincreasing in $r$ for $r > 0)$, and the positive constants $\Gamma_1^*$, $\Gamma_2^*$, $\beta$, $N$, such that:

1) $\Gamma_1^*, \Gamma_2^*$ and $\beta$ satisfy conditions 1° and 4° of Theorems 1 and 2 and condition 4° is a strict inequality

2) $u_1(\| D(x(\theta),\ t) \|,\ t) \leqslant V(x(\theta),\ t) \leqslant u_2(\| x(\theta) \|,\ t)$

$$\bar{V}_1^{\cdot} \leqslant u_3(\| D(x(\theta),\ t) \|,\ t), \quad \forall\ t \in [t_0,\ t_0 + T)$$

3) $\displaystyle\int_{t_1}^{t_2} u_3(0, t)\,dt < u_1(\Gamma_1^*, t_2) - u_2(\alpha,\ t_1), \quad \forall t_1, t_2 \in [t_0, t_0 + T); \quad t_2 > t_1$

4) $\displaystyle\int_{t_1}^{t_2} u_3(0, t)\,dt < u_1(\Gamma_2^*, t_2) - u_2(\beta, t_1), \quad \forall t_1, t_2 \in [t_0, t_0 + T); \quad t_2 > t_1$

5) $\|f(x(\theta),\ t)\| \leqslant N,\quad \|x(\theta)\| < A;\quad t \in [t_0,\ t_0 + T)$

If, furthermore, for some integer $k$ $(1 \leqslant k \leqslant k_0)$, where $k_0$ is an integer solution of the inequality $k_0\nu < T \leqslant (k_0 + 1)\nu,\ \nu \equiv a^{*-1}\ln[(K_1{}^*\alpha + K_2{}^*\Gamma_1{}^*)\beta^{-1}]$ there exists a partitioning of the interval $[t_0,\ t_0 + T]$ by points $t_j$

6) $t_{j-1} + \nu \leqslant t_j \leqslant t_0 + T - (k - j)\nu,\quad j = 1, 2, \ldots, k-1$

such that for the constants

$$\beta_{kj} \equiv K_3{}^{*^{-1}}[\beta - (K_1{}^*\alpha + K_2{}^*\Gamma_1{}^*)\exp\{-a^*(t_j - t_{j-1}\}]$$
$$\pi_{kj} \equiv \max[\beta_{kj} - N(t_j - t_{j-1});\ 0],\quad j = 1,\ 2,\ldots,k$$
$$t_k \equiv t_0 + T$$

we can select a collection of numbers $\rho_{kj},\ j = 1,\ 2,\ldots,k$ $(\pi_{kj} \leqslant \rho_{kj} \leqslant \beta_{kj})$, for which

7) $$\sum_{j=1}^{k} N^{-1}[\beta_{kj} - \rho_{kj}] \sup_{t_{j-1} \leqslant t \leqslant t_j} u_3(\rho_{kj}, t) < u_1(\pi_{kk}, t_0 + T) - u_2(\alpha, t_0)$$

then system (1.1) is $(\alpha,\ A,\ B,\ t_0,\ T)$-contractively stable.

Proof. From relations (3) and (4) and condition 1° of Theorem 1 it follows that the solution $x_t(\theta)$ defined by the initial function $\beta \leqslant \|x_{t_0}(\theta)\| \leqslant \alpha$ satisfies the conditions $\|D(x_t(\theta),\ t)\| < \Gamma_1{}^*$ and $\|x_t(\theta)\| < A$ for all $t \in [t_0,\ t_0 + T)$.

Let us now assume that $\|x_t(\theta)\| > \beta$ for all $t \in [t_0,\ t_0 + T)$. We partition the interval $[t_0,\ t_0 + T]$ by $k$ points whose abscissas satisfy condition (6). Setting $s = t_{j-1}$ and $t = t_j$, from relation (2.6) we have

$$\beta \leqslant [K_1{}^*\alpha + K_2{}^* \sup_{t_0 \leqslant u \leqslant t_j} \|D(x_u(\theta), u)\|]\exp\{-a^*(t_j - t_{j-1})\} +$$
$$K_3{}^*\ T_j \leqslant (K_1{}^*\alpha + K_2{}^*\Gamma_1{}^*)\exp\{-a^*(t_j - t_{j-1})\} + K_3{}^*T_j$$
$$T_j \equiv \sup_{t_{j-1} \leqslant u \leqslant t_j} \|D(x_u(\theta), u)\|$$

whence $T_j \geqslant \beta_{kj}$. The latter implies the existence of an instant $t_j{}^\circ \in [t_{j-1},\ t_j]$ at which

$$\|D(x_{t_j{}^\circ}(\theta),\ t_j{}^\circ)\| \geqslant \beta_{kj}$$

Using the properties of function $u_3\ (r,\ t)$ and the inequality obtained, on each of the $k$ segments being examined, we have

$$\int_{t_{j-1}}^{t_j} \bar{V}_1{}^{\boldsymbol{\cdot}}(t)\,dt \leqslant N^{-1}U_j,\quad U_j \equiv [\beta_{kj} - \rho_{kj}] \sup_{t_{j-1} \leqslant t \leqslant t_j} u_3(\rho_{kj}, t)$$

for any $\rho_{kj}$ $(\pi_{kj} \leqslant \rho_{kj} \leqslant \beta_{kj})$. The estimate

$$\int_{t_0}^{t_0+T} \bar{V}_1{}^{\boldsymbol{\cdot}}(t)\,dt \leqslant N^{-1}\sum_{j=1}^{k} U_j \tag{3.4}$$

is valid on the interval $[t_0,\ t_0 + T]$.

Let us assume that the interval partitioning being examined and the selected collection of constants $\rho_{kj}$ satisfy the hypotheses of Corollary 1. Then, comparing relation (3.4) and conditions (3) and (4), we infer the existence of an instant $t^* \in (t_0,\ t_0 + T)$ for which $\|x_{t^*}(\theta)\| < \beta$. Now, with the aid of conditions (3) and (4) of Corollary 1 and of condition 4° of Theorem 2 it is not difficult to show that $\| x_t(\theta) \| < B$ for $t \in [t^*, t_0 + T)$. The proof is obvious when the initial function $x_{t_0}(\theta)$ belongs to the domain $x(\theta) \| < \beta$. The corollary is proved.

**4.** Let us now derive the conditions for the $(\alpha, A, t_0, T)$-instability of system (1.1).

Theorem 3. If a functional $V(x(\theta), t)$, a bounded function $\zeta(t)$ and an integrable function $\psi(t)$ exist such that the conditions

$$1^\circ.\ \underline{V}_1^{\boldsymbol{\cdot}} \geqslant \psi(t), \quad x(\theta) \in Q(t), \quad \forall\, t \in [t_0, t_0 + T_1)$$

$$2^\circ.\ \int_{t_0}^{t_1} \psi(t)\, dt \geqslant \zeta(t_1) - \zeta(t_0), \quad \forall t_1 \in [t_0, t_0 + T_1)$$

$$3^\circ.\ \int_{t_0}^{t_0+T_1} \psi(t)\, dt \geqslant \sup_{Q(t_0+T_1)} V(x(\theta), t_0 + T_1) - \zeta(t_0)$$

are fulfilled in a nonempty connected set $Q(t)$ defined by the relations ($T_1$ is some constant, $0 < T_1 \leqslant T$)

a) $Q(t) = Q_1(t) \cap Q_2(t), \quad \forall\, t \in [t_0, t_0 + T_1)$

$$Q_1(t) = \{x(\theta):\ \|x(\theta)\| \leqslant A,\ \ \|D(x(\theta), t)\| \leqslant L(t) \|x(\theta)\|\}$$

$$Q_2(t) = \{x(\theta):\ V(x(\theta), t) > \zeta(t)\}$$

b) $Q(t_0) \cap \{\dot{x}(\theta):\ \|x(\theta)\| \leqslant \alpha\} \neq \varnothing$

c) $\exists t^* \in (t_0, t_0 + T_1), \quad Q(t^*) \cap \{x(\theta):\ \|x(\theta)\| = A\} \neq \varnothing$

then system (1.1) is $(\alpha, A, t_0, T)$-unstable.

Proof. Let $x_t(\theta)$ be a solution of system (1.1), defined by the initial function $x_{t_0}(\theta) \in Q(t_0)$, $\| x_{t_0}(\theta) \| \leqslant \alpha$. By the theorem's hypothesis, $V(x_{t_0}(\theta), t_0) > \zeta(t_0)$. We assume that $V(x_{t_1}(\theta), t_1) = \zeta(t_1)$ for the first time at the instant $t_1 \in (t_0, t_0 + T_1)$. Here it is natural to assume that $\| x_t(\theta) \| < A$ for all $t \in [t_0, t_0 + T_1)$. Then

$$\zeta(t_1) - \zeta(t_0) > V(x_{t_1}(\theta), t_1) - V(x_{t_0}(\theta), t_0) \geqslant$$

$$\int_{t_0}^{t_1} \underline{V}_1^{\boldsymbol{\cdot}}(t)\, dt \geqslant \int_{t_0}^{t_1} \psi(t)\, dt \geqslant \zeta(t_1) - \zeta(t_0)$$

The relation obtained is a contradiction. Therefore, $V(x_t(\theta), t) > \zeta(t)$ for all $t \in [t_0, t_0 + T_1)$ and, consequently, $x_t(\theta) \in Q(t)$. Using this fact, from 3° we obtain the contradictory inequality

$$\sup_{Q(t_0+T_1)} V(x(\theta), t_0 + T_1) - \zeta(t_0) > V(x_{t_0+T_1}(\theta), t_0 + T_1) - V(x_{t_0}(\theta), t_0) \geqslant$$

$$\int_{t_0}^{t_0+T_1} \underline{V}_1^{\boldsymbol{\cdot}}(t)\, dt \geqslant \int_{t_0}^{t_0+T_1} \psi(t)\, dt \geqslant \sup_{Q(t_0+T_1)} V(x(\theta), t_0 + T_1) - \zeta(t_0)$$

Hence, an instant $t_2 \in (t_0, t_0 + T_1)$ such that $\| x_{t_2}(\theta) \| = A$ exists. The theorem is proved.

Corollary 2. If a continuous function $u_1(r_1, r_2, t)$, increasing in $r_2$ for $r_2 > 0$, the continuous functions $u_i(r, t)$, $i = 2, 3$, increasing in $r$ for $r > 0$, the positive functions $\beta(t)$ and $\gamma(t)$ and the constant $T_1$ $(0 < T_1 \leqslant T)$ exist such that

1°. $u_1(\|x(\theta)\|; \|D(x(\theta), t)\|; t) \leqslant V(x(\theta), t) \leqslant u_2(\|D(x, (\theta), t)\|; t)$

$\underline{V}_1^{\bullet} \geqslant u_3(\|D(x(\theta), t)\|, t)$

2°. $\beta(t) < \alpha, \quad \gamma(t) < L(t)\beta(t)$

3°. $u_1^{-1}[r_1; u_1(\beta(t), \gamma(t), t); t] < L(t) r_1 \quad$ for $\beta(t) < r_1 \leqslant A$

4°. $\int\limits_{t_0}^{t} u_3\{u_2^{-1}[u_1(\beta(s), \gamma(s), s) s], s\} ds \geqslant u_1(\beta(t), \gamma(t), t) - u_1(\beta(t_0), \gamma(t_0), t_0)$

5°. $\int\limits_{t_0}^{t_0+T_1} u_3\{u_2^{-1}[u_1(\beta(s), \gamma(s), s), s], s\} ds \geqslant u_2[AL(t_0+T_1), t_0+T_1) -$

$- u_1(\beta(t_0), \gamma(t_0), t_0)$

for all $t \in [t_0, t_0 + T_1)$, then system (1.1) is $(\alpha, A, t_0, T)$-unstable.

For the proof it is sufficient to consider the function $u_1(\beta(t), \gamma(t), t)$ as $\zeta(t)$ and to construct the required domain $Q(t)$ with the aid of conditions 3° and 4°.

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